6.1 Ratios, Proportions, and the Geometric Mean

Before You solved problems by writing and solving equations.

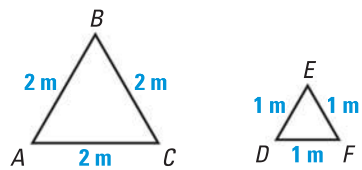
Now You will solve problems by writing and solving proportions.

Why? So you can estimate bird populations, as in Ex. 62.

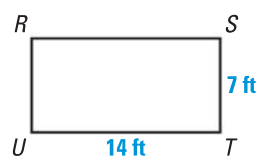
Key Vocabulary
- **ratio**
- **proportion**
  means, extremes
- **geometric mean**

If a and b are two numbers or quantities and $b \neq 0$, then the **ratio of *a* to *b*** is $\frac{a}{b}$. The ratio of a to b can also be written as $a:b$.

For example, the ratio of a side length in $\triangle ABC$ to a side length in $\triangle DEF$ can be written as $\frac{2}{1}$ or $2:1$.



Ratios are usually expressed in simplest form. Two ratios that have the same simplified form are called *equivalent ratios*. The ratios 7 : 14 and 1 : 2 in the example below are *equivalent*.



$$\frac{\text{width of } RSTU}{\text{length of } RSTU} = \frac{7 \text{ ft}}{14 \text{ ft}} = \frac{1}{2}$$

EXAMPLE 1 Simplify ratios

Simplify the ratio.

a. 64 m : 6 m

b. $\frac{5 \text{ ft}}{20 \text{ in.}}$

Solution

a. Write 64 m : 6 m as $\frac{64 \text{ m}}{6 \text{ m}}$. Then divide out the units and simplify.

$$\frac{64 \text{ m}}{6 \text{ m}} = \frac{32}{3} = 32:3$$

b. To simplify a ratio with unlike units, multiply by a conversion factor.

$$\frac{5 \text{ ft}}{20 \text{ in.}} = \frac{5 \text{ ft}}{20 \text{ in.}} \cdot \frac{12 \text{ in.}}{1 \text{ ft}} = \frac{60}{20} = \frac{3}{1}$$

REVIEW UNIT ANALYSIS

For help with measures and conversion factors, see p. 886 and the Table of Measures on p. 921.

✓ GUIDED PRACTICE for Example 1

Simplify the ratio.

1. 24 yards to 3 yards

2. 150 cm : 6 m

EXAMPLE 2 Use a ratio to find a dimension

PAINTING You are planning to paint a mural on a rectangular wall. You know that the perimeter of the wall is 484 feet and that the ratio of its length to its width is 9 : 2. Find the area of the wall.

Solution

WRITE EXPRESSIONS

Because the ratio in Example 2 is 9 : 2, you can write an equivalent ratio to find expressions for the length and width.

$$\frac{\text{length}}{\text{width}} = \frac{9}{2} = \frac{9}{2} \cdot \frac{x}{x} = \frac{9x}{2x}.$$

STEP 1 **Write** expressions for the length and width. Because the ratio of length to width is 9 : 2, you can represent the length by $9x$ and the width by $2x$.

STEP 2 **Solve** an equation to find x.

$2\ell + 2w = P$	Formula for perimeter of rectangle
$2(9x) + 2(2x) = 484$	Substitute for ℓ, w, and P.
$22x = 484$	Multiply and combine like terms.
$x = 22$	Divide each side by 22.

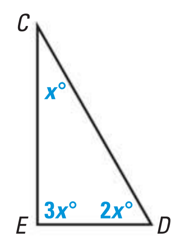
STEP 3 **Evaluate** the expressions for the length and width. Substitute the value of x into each expression.

Length $= 9x = 9(22) = 198$ Width $= 2x = 2(22) = 44$

▶ The wall is 198 feet long and 44 feet wide, so its area is 198 ft • 44 ft = 8712 ft^2.

EXAMPLE 3 Use extended ratios

xy ALGEBRA The measures of the angles in △*CDE* are in the *extended ratio* of 1 : 2 : 3. Find the measures of the angles.



Solution

Begin by sketching the triangle. Then use the extended ratio of 1 : 2 : 3 to label the measures as $x°$, $2x°$, and $3x°$.

$x° + 2x° + 3x° = 180°$	Triangle Sum Theorem
$6x = 180$	Combine like terms.
$x = 30$	Divide each side by 6.

▶ The angle measures are 30°, 2(30°) = 60°, and 3(30°) = 90°.

✓ GUIDED PRACTICE for Examples 2 and 3

3. The perimeter of a room is 48 feet and the ratio of its length to its width is 7 : 5. Find the length and width of the room.

4. A triangle's angle measures are in the extended ratio of 1 : 3 : 5. Find the measures of the angles.

PROPORTIONS An equation that states that two ratios are equal is called a **proportion**.

$$\text{extreme} \rightarrow \frac{a}{b} = \frac{c}{d} \leftarrow \text{mean}$$
$$\text{mean} \rightarrow \qquad \leftarrow \text{extreme}$$

The numbers b and c are the **means** of the proportion. The numbers a and d are the **extremes** of the proportion.

The property below can be used to solve proportions. To *solve a proportion*, you find the value of any variable in the proportion.

PROPORTIONS
You will learn more properties of proportions on p. 364.

KEY CONCEPT *For Your Notebook*

A Property of Proportions

1. **Cross Products Property** In a proportion, the product of the extremes equals the product of the means.

If $\frac{a}{b} = \frac{c}{d}$ where $b \neq 0$ and $d \neq 0$, then $ad = bc$.

$\frac{2}{3} = \frac{4}{6}$ $\quad 3 \cdot 4 = 12$
$\quad 2 \cdot 6 = 12$

EXAMPLE 4 Solve proportions

ALGEBRA **Solve the proportion.**

a. $\frac{5}{10} = \frac{x}{16}$ **b.** $\frac{1}{y+1} = \frac{2}{3y}$

Solution

a. $\frac{5}{10} = \frac{x}{16}$ Write original proportion.

$5 \cdot 16 = 10 \cdot x$ Cross Products Property

$80 = 10x$ Multiply.

$8 = x$ Divide each side by 10.

b. $\frac{1}{y+1} = \frac{2}{3y}$ Write original proportion.

$1 \cdot 3y = 2(y+1)$ Cross Products Property

$3y = 2y + 2$ Distributive Property

$y = 2$ Subtract $2y$ from each side.

ANOTHER WAY
In part (a), you could multiply each side by the denominator, 16.
Then $16 \cdot \frac{5}{10} = 16 \cdot \frac{x}{16}$, so $8 = x$.

GUIDED PRACTICE for Example 4

Solve the proportion.

5. $\frac{2}{x} = \frac{5}{8}$ **6.** $\frac{1}{x-3} = \frac{4}{3x}$ **7.** $\frac{y-3}{7} = \frac{y}{14}$

EXAMPLE 5 Solve a real-world problem

SCIENCE As part of an environmental study, you need to estimate the number of trees in a 150 acre area. You count 270 trees in a 2 acre area and you notice that the trees seem to be evenly distributed. Estimate the total number of trees.

Solution

Write and solve a proportion involving two ratios that compare the number of trees with the area of the land.

$\frac{270}{2} = \frac{n}{150}$ ← number of trees / ← area in acres	**Write proportion.**
$270 \cdot 150 = 2 \cdot n$	**Cross Products Property**
$20{,}250 = n$	**Simplify.**

▶ There are about 20,250 trees in the 150 acre area.

KEY CONCEPT — *For Your Notebook*

Geometric Mean

The **geometric mean** of two positive numbers a and b is the positive number x that satisfies $\frac{a}{x} = \frac{x}{b}$. So, $x^2 = ab$ and $x = \sqrt{ab}$.

EXAMPLE 6 Find a geometric mean

Find the geometric mean of 24 and 48.

Solution

$x = \sqrt{ab}$	**Definition of geometric mean**
$= \sqrt{24 \cdot 48}$	**Substitute 24 for *a* and 48 for *b*.**
$= \sqrt{24 \cdot 24 \cdot 2}$	**Factor.**
$= 24\sqrt{2}$	**Simplify.**

▶ The geometric mean of 24 and 48 is $24\sqrt{2} \approx 33.9$.

✓ GUIDED PRACTICE for Examples 5 and 6

8. **WHAT IF?** In Example 5, suppose you count 390 trees in a 3 acre area of the 150 acre area. Make a new estimate of the total number of trees.

Find the geometric mean of the two numbers.

9. 12 and 27 **10.** 18 and 54 **11.** 16 and 18

6.1 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 5, 27, and 59

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 47, 48, 52, and 63

◆ = **MULTIPLE REPRESENTATIONS** Ex. 66

SKILL PRACTICE

1. **VOCABULARY** Copy the proportion $\frac{m}{n} = \frac{p}{q}$. Identify the means of the proportion and the extremes of the proportion.

2. ★ **WRITING** Write three ratios that are equivalent to the ratio 3 : 4. *Explain* how you found the ratios.

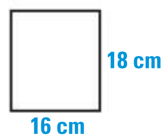
EXAMPLE 1 on p. 356 for Exs. 3–17

SIMPLIFYING RATIOS **Simplify the ratio.**

3. \$20 : \$5
4. $\frac{15\text{ cm}^2}{12\text{ cm}^2}$
5. 6 L : 10 mL
6. $\frac{1\text{ mi}}{20\text{ ft}}$
7. $\frac{7\text{ ft}}{12\text{ in.}}$
8. $\frac{80\text{ cm}}{2\text{ m}}$
9. $\frac{3\text{ lb}}{10\text{ oz}}$
10. $\frac{2\text{ gallons}}{18\text{ quarts}}$

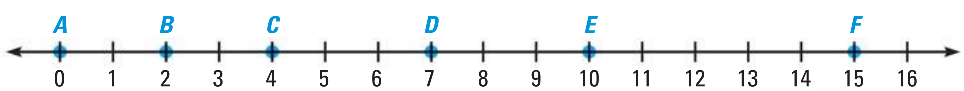
WRITING RATIOS **Find the ratio of the width to the length of the rectangle. Then simplify the ratio.**

11. 

12. 

13. 

FINDING RATIOS **Use the number line to find the ratio of the distances.**

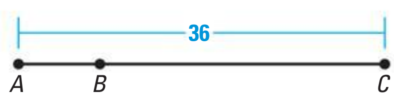


14. $\frac{AD}{CF}$
15. $\frac{BD}{AB}$
16. $\frac{CE}{EF}$
17. $\frac{BE}{CE}$

EXAMPLE 2 on p. 357 for Exs. 18–19

18. **PERIMETER** The perimeter of a rectangle is 154 feet. The ratio of the length to the width is 10 : 1. Find the length and the width.

19. **SEGMENT LENGTHS** In the diagram, $AB : BC$ is 2 : 7 and $AC = 36$. Find AB and BC.



EXAMPLE 3 on p. 357 for Exs. 20–22

USING EXTENDED RATIOS **The measures of the angles of a triangle are in the extended ratio given. Find the measures of the angles of the triangle.**

20. 3 : 5 : 10
21. 2 : 7 : 9
22. 11 : 12 : 13

EXAMPLE 4 on p. 358 for Exs. 23–30

xy **ALGEBRA** **Solve the proportion.**

23. $\frac{6}{x} = \frac{3}{2}$
24. $\frac{y}{20} = \frac{3}{10}$
25. $\frac{2}{7} = \frac{12}{z}$
26. $\frac{j+1}{5} = \frac{4}{10}$
27. $\frac{1}{c+5} = \frac{3}{24}$
28. $\frac{4}{a-3} = \frac{2}{5}$
29. $\frac{1+3b}{4} = \frac{5}{2}$
30. $\frac{3}{2p+5} = \frac{1}{9p}$

EXAMPLE 6 on p. 359 for Exs. 31–36

GEOMETRIC MEAN **Find the geometric mean of the two numbers.**

31. 2 and 18 **32.** 4 and 25 **33.** 32 and 8

34. 4 and 16 **35.** 2 and 25 **36.** 6 and 20

37. **ERROR ANALYSIS** A student incorrectly simplified the ratio. *Describe* and correct the student's error.

$$\frac{8 \text{ in.}}{3 \text{ ft}} = \frac{8 \text{ in.}}{3 \text{ ft}} \cdot \frac{12 \text{ in.}}{1 \text{ ft}} = \frac{96 \text{ in.}}{3 \text{ ft}} = \frac{32 \text{ in.}}{1 \text{ ft}}$$

WRITING RATIOS **Let $x = 10$, $y = 3$, and $z = 8$. Write the ratio in simplest form.**

38. $x:z$ **39.** $\frac{8y}{x}$ **40.** $\frac{4}{2x + 2z}$ **41.** $\frac{2x - z}{3y}$

ALGEBRA **Solve the proportion.**

42. $\frac{2x + 5}{3} = \frac{x - 5}{4}$ **43.** $\frac{2 - s}{3} = \frac{2s + 1}{5}$ **44.** $\frac{15}{m} = \frac{m}{5}$ **45.** $\frac{7}{q + 1} = \frac{q - 1}{5}$

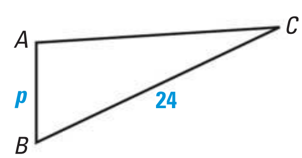
46. **ANGLE MEASURES** The ratio of the measures of two supplementary angles is 5 : 3. Find the measures of the angles.

47. ★ **SHORT RESPONSE** The ratio of the measure of an exterior angle of a triangle to the measure of the adjacent interior angle is 1 : 4. Is the triangle *acute* or *obtuse*? *Explain* how you found your answer.

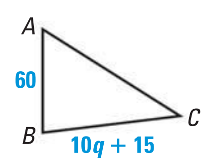
48. ★ **SHORT RESPONSE** Without knowing its side lengths, can you determine the ratio of the perimeter of a square to the length of one of its sides? *Explain.*

ALGEBRA **In Exercises 49–51, the ratio of two side lengths for the triangle is given. Solve for the variable.**

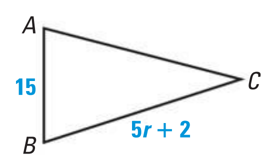
49. $AB:BC$ is 3 : 8.



50. $AB:BC$ is 3 : 4.



51. $AB:BC$ is 5 : 9.



52. ★ **MULTIPLE CHOICE** What is a value of x that makes $\frac{x}{3} = \frac{4x}{x + 3}$ true?

Ⓐ 3 Ⓑ 4 Ⓒ 9 Ⓓ 12

53. **AREA** The area of a rectangle is 4320 square inches. The ratio of the width to the length is 5 : 6. Find the length and the width.

54. **COORDINATE GEOMETRY** The points $(-3, 2)$, $(1, 1)$, and $(x, 0)$ are collinear. Use slopes to write a proportion to find the value of x.

55. **ALGEBRA** Use the proportions $\frac{a + b}{2a - b} = \frac{5}{4}$ and $\frac{b}{a + 9} = \frac{5}{9}$ to find a and b.

56. **CHALLENGE** Find the ratio of x to y given that $\frac{5}{y} + \frac{7}{x} = 24$ and $\frac{12}{y} + \frac{2}{x} = 24$.

PROBLEM SOLVING

EXAMPLE 2 on p. 357 for Ex. 57

57. TILING The perimeter of a room is 66 feet. The ratio of its length to its width is 6 : 5. You want to tile the floor with 12 inch square tiles. Find the length and width of the room, and the area of the floor. How many tiles will you need? The tiles cost $1.98 each. What is the total cost to tile the floor?

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58. GEARS The *gear ratio* of two gears is the ratio of the number of teeth of the larger gear to the number of teeth of the smaller gear. In a set of three gears, the ratio of Gear A to Gear B is equal to the ratio of Gear B to Gear C. Gear A has 36 teeth and Gear C has 16 teeth. How many teeth does Gear B have?

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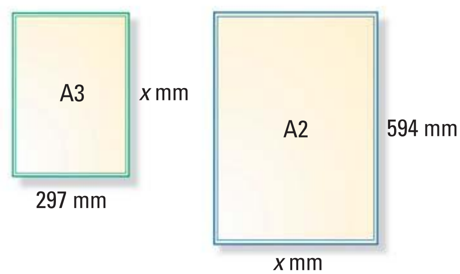
59. TRAIL MIX You need to make 36 one-half cup bags of trail mix for a class trip. The recipe calls for peanuts, chocolate chips, and raisins in the extended ratio 5 : 1 : 4. How many cups of each item do you need?

60. PAPER SIZES International standard paper sizes are commonly used all over the world. The various sizes all have the same width-to-length ratios. Two sizes of paper are shown, called A3 and A2. The distance labeled x is the geometric mean of 297 mm and 594 mm. Find the value of x.



61. BATTING AVERAGE The batting average of a baseball player is the ratio of the number of hits to the number of official at-bats. In 2004, Johnny Damon of the Boston Red Sox had 621 official at-bats and a batting average of .304. Use the proportion to find the number of hits made by Johnny Damon.

$$\frac{\text{Number of hits}}{\text{Number of at-bats}} = \frac{\text{Batting average}}{1.000}$$

EXAMPLE 5 on p. 359 for Ex. 62

62. MULTI-STEP PROBLEM The population of Red-tailed hawks is increasing in many areas of the United States. One long-term survey of bird populations suggests that the Red-tailed hawk population is increasing nationally by 2.7% each year.

a. Write 2.7% as a ratio of hawks in year n to hawks in year $(n - 1)$.

b. In 2004, observers in Corpus Christi, TX, spotted 180 migrating Red-tailed hawks. Assuming this population follows the national trend, about how many Red-tailed hawks can they expect to see in 2005?

c. Observers in Lipan Point, AZ, spotted 951 migrating Red-tailed hawks in 2004. Assuming this population follows the national trend, about how many Red-tailed hawks can they expect to see in 2006?

63. ★ **SHORT RESPONSE** Some common computer screen resolutions are 1024 : 768, 800 : 600, and 640 : 480. *Explain* why these ratios are equivalent.

64. **BIOLOGY** The larvae of the Mother-of-Pearl moth is the fastest moving caterpillar. It can run at a speed of 15 inches per second. When threatened, it can curl itself up and roll away 40 times faster than it can run. How fast can it run in miles per hour? How fast can it roll?

65. **CURRENCY EXCHANGE** Emily took 500 U.S. dollars to the bank to exchange for Canadian dollars. The exchange rate on that day was 1.2 Canadian dollars per U.S. dollar. How many Canadian dollars did she get in exchange for the 500 U.S. dollars?

66. **MULTIPLE REPRESENTATIONS** Let x and y be two positive numbers whose geometric mean is 6.

   a. **Making a Table** Make a table of ordered pairs (x, y) such that $\sqrt{xy} = 6$.

   b. **Drawing a Graph** Use the ordered pairs to make a scatter plot. Connect the points with a smooth curve.

   c. **Analyzing Data** Is the data linear? Why or why not?

67. **xy ALGEBRA** Use algebra to verify Property 1, the Cross Products Property.

68. **xy ALGEBRA** Show that the geometric mean of two numbers is equal to the arithmetic mean (or average) of the two numbers only when the numbers are equal. (*Hint*: Solve $\sqrt{xy} = \frac{x+y}{2}$ with $x, y \geq 0$.)

CHALLENGE In Exercises 69–71, use the given information to find the value(s) of *x*. Assume that the given quantities are nonnegative.

69. The geometric mean of the quantities $(\sqrt{x})$ and $(3\sqrt{x})$ is $(x - 6)$.

70. The geometric mean of the quantities $(x + 1)$ and $(2x + 3)$ is $(x + 3)$.

71. The geometric mean of the quantities $(2x + 1)$ and $(6x + 1)$ is $(4x - 1)$.

MIXED REVIEW

PREVIEW
Prepare for Lesson 6.2 in Exs. 72–75.

Find the reciprocal. *(p. 869)*

72. -6 73. $\frac{1}{13}$ 74. $\frac{-36}{3}$ 75. -0.2

Solve the quadratic equation. *(p. 882)*

76. $5x^2 = 35$ 77. $x^2 - 20 = 29$ 78. $(x - 3)(x + 3) = 27$

Write the equation of the line with the given description. *(p. 180)*

79. Parallel to $y = 3x - 7$, passing through $(1, 2)$

80. Perpendicular to $y = \frac{1}{4}x + 5$, passing through $(0, 24)$